Experiment ???
Fourier analysis, beats and Doppler shift

1 Introduction

This laboratory exercise will deal with three somewhat related topics, Fourier analysis, beats and the Doppler shift. Fourier analysis is a mathematical tool that allows us to extract information about the frequency spectrum of any signal. We’re going to apply it to the other two topics.

“Beats” refers to the situation where two periodic signals, having nearly the same frequencies, interfere. Since the frequencies are not equal, the signals are, at times, in phase and, at other times, out of phase. The resulting superposed signal has amplitude which alternates between large and small, depending on the relative phases of the two signals.

Most of us have experienced the Doppler effect, probably in a situation where a car or motorcycle passes us at a relatively high speed. The sound, specifically the pitch or frequency, of the engine changes as the vehicle passes us. The vehicle begins by approaching us, but after it has passed, it is receding from us. This characteristic rise in frequency as the vehicle approaches and fall in frequency as the vehicle recedes is called the Doppler effect. The Doppler effect occurs any time either the source, the observer or both are in motion. In this experiment, we’re going to study the Doppler effect using a moving source; the moving observer version is similar.

2 Theory

2.1 Fourier analysis

Fourier analysis is a mathematical tool for extracting knowledge of the underlying frequency spectrum in any signal. It is related to the idea of the Fourier series, with which any repeating waveform can be expressed in generally an infinite sum of sine and cosine functions. Here is an example of a partial sum for 1 term and 6 terms.
The mathematical definition of the Fourier series is
\[ f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left( a_n \cos\left(\frac{n\pi t}{T}\right) + b_n \sin\left(\frac{n\pi t}{T}\right) \right) \]  
\[ a_n = \frac{1}{T} \int_{-T}^{T} f(t) \cos\left(\frac{n\pi t}{T}\right) dt \]  
\[ b_n = \frac{1}{T} \int_{-T}^{T} f(t) \sin\left(\frac{n\pi t}{T}\right) dt. \]

The coefficients \( a_n \) and \( b_n \) tell the relative importance of the frequency \( \frac{n\pi}{T} \). The Fourier decomposition works because
\[ \int_{-T}^{T} \cos\left(\frac{m\pi t}{T}\right) \cos\left(\frac{n\pi t}{T}\right) dt = 0 \]  
\[ \int_{-T}^{T} \sin\left(\frac{m\pi t}{T}\right) \sin\left(\frac{n\pi t}{T}\right) dt = 0 \]
unless \( m = n \) and
\[ \int_{-T}^{T} \cos\left(\frac{m\pi t}{T}\right) \sin\left(\frac{n\pi t}{T}\right) dt = 0 \]
always.

The Fourier series can be generalized to a situation where the signal doesn’t necessarily repeat, for example a short segment of human speech or a short sound produced by a musical instrument. This is the Fourier transform, and it is defined as
\[ f(t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} g(\omega) \exp^{i\omega t} \]  
where
\[ g(\omega) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} f(t) \exp^{-i\omega t}. \]

Notice that \( g \) and \( f \) are closely related to each other; \( g(\omega) \) is called the Fourier transform of \( f(t) \), where \( \omega = 2\pi f \). As with the Fourier series, \( g(\omega) \) tells about the relative strength of a pure, sinusoidal signal with frequency \( f = \frac{\omega}{2\pi} \). The closer a signal is to being a pure sinusoidal signal, the narrower \( g(\omega) \) becomes.

Although we can’t, in general, calculate the Fourier transform of experimental data taken in lab, various mathematics packages are available which can, to a very good approximation, take the Fourier transform of any data we may have. The software package Excel has an analysis package which will allow us to take the Fourier transform of data, and extract the frequency spectrum.

### 2.2 Beats

Consider two sinusoidal signals:
\[ S_1(t) = A \cos(2\pi f_1 t) \]  
\[ S_2(t) = A \cos(2\pi f_2 t) \]
which are added together. The result is
\[ S(t) = 2A \cos(2\pi \bar{f} t) \cos\left(2\pi \frac{f_{\text{beat}}}{2} t\right) \]
where \( \bar{f} = \frac{f_1 + f_2}{2} \) and \( f_{\text{beat}} = |f_1 - f_2| \). This is illustrated in the figure below. You verified this result by algebraically in the pre-lab.
The perceived frequency is given by the average of the two original frequencies, $\bar{f}$, as defined above. $\bar{f}$ can be determined by examining the fast oscillations in the summed signal: the number of "wiggles" in a certain time, divided by the time will give $\bar{f}$. This part of the summed signal is due to the $\cos(2\pi \bar{f} t)$. The beat frequency $f_{\text{beat}}$ can be determined from the envelope, or the variation in amplitude with time, given by the $2A\cos(2\pi \frac{f_{\text{beat}}}{2} t)$ part above. Thus, given a signal which is the sum of two pure sinusoidal signals with equal amplitudes and different frequencies, you can extract the original frequencies by making some simple measurements on the waveform.

As described above, you can also Fourier analyze the signal, and you should see two strong peaks at the two original frequencies. This will be the first experimental task.

2.3 Doppler effect

2.3.1 General

Sound waves leaving a point source (S) expand in spherical waves centered on the source. Typically, such waves are are represented by concentric circles, where the circles correspond to the wavefronts (high pressure regions), as indicated in the figure:
The observer detects sound waves of wavelength $\lambda$ (the distance between circles) and the same frequency as emitted by the source:

$$\lambda = vT$$
$$\lambda = v/f$$
$$\lambda f = v,$$  \hspace{1cm} (8)

where $T$ is the period of the sound wave, and $v$ is the speed of sound.

If the observer ($O$) is moving towards the source ($S$), the wavefronts reach the observer more frequently, as the observer is moving towards the wavefronts. The distance between the wavefronts has not changed, so

$$\lambda = (v + v_o)T'$$
$$\lambda = (v + v_o)/f'$$
$$f' = (v + v_o)/\lambda$$
$$f' = \frac{v + v_o}{v}$$  \hspace{1cm} (9)

where $f'$ is the detected frequency. If the observer is moving away from the source, the sign on $v_o$ changes, giving the general result for a moving observer:

$$f' = \frac{v \pm v_o}{v}$$  \hspace{1cm} (10)

where the upper sign corresponds to an approaching observer and the lower sign to a receding observer.

If, instead, the source ($S$) is moving and the observer is stationary, there is a change in the apparent wavelength of the sound waves, and its value depends on where the observer is positioned:

If the source is moving directly towards the observer as pictured, the distance between adjacent wavefronts is $\lambda' = (v - v_s)T$ and we can relate the new and original frequencies:

$$\lambda' = (v - v_s)T$$
$$v/f' = (v - v_s)/f$$
$$f' = f \frac{v}{v - v_s}.$$  \hspace{1cm} (11)
Again, if the source is moving directly away from the observer, it is easy to see that the sign on $v_s$ is simply changed, giving the general result for a moving source:

$$f' = f \frac{v}{(v \mp v_s)}$$  \hspace{1cm} (12)

where, the upper sign corresponds to the source approaching the observer and the lower sign corresponds to the source receding from the observer.

Finally, it’s possible to generalize to the case where both the source and observer are moving:

$$f' = f \frac{(v \pm v_o)}{(v \mp v_s)}.$$  \hspace{1cm} (13)

The signs above are independent of each other! Use an upper sign for the source or observer approaching the other and a lower sign for the source or observer receding from the other. As an example, consider the source being a police car, heading east at 100 km/h and the observer is in another car also heading east at 100 km/h, trying to escape the from the police. In this example, the source is approaching the observer (upper sign) while the observer is receding from the source (lower sign) and the detected frequency is

$$f' = f \frac{(v - v_o)}{(v + v_s)}$$

$$f' = f.$$  \hspace{1cm} (14)

2.3.2 Qualitative and Quantitative Understanding

The pre-lab tested your understanding of the Doppler shift, both qualitatively and quantitatively. You should discuss your understanding of the Doppler shift with fellow students and with your instructor before continuing on to the experimental task.
3 EXPERIMENTAL TASK

3.1 Beats

Apparatus

The apparatus needed for this experiment is a computer with a Pasco Science Workshop interface attached and a sound sensor. In addition, a dual channel function generator connected to a speaker will provide the beat sample.

Procedure

1) Connect the sound sensor to the Science Workshop interface box, and set the sampling options as suggested by your instructor. Your instructor will tell you the approximate frequency being output by the function generator. Listen to the beats, and estimate the beat frequency by ear (count the beats for some period of time, 10 or 15 seconds should be sufficient, and divide the number of beats by the time). Record the approximate frequency and your estimated beat frequency.

2) Record a few second sample of the sound. Open a graph window, and determine the average frequency and beat frequency, using the data recorded by the computer. This procedure is discussed in the Theory section above.

3) Export the data to be analyzed using Excel. Open a table window, which will show the numerical values for the data displayed in the graph. Click on the small clock icon in the table window, in order to display the time and amplitude of the wave. If necessary, adjust the number of digits displayed, by clicking on the button labeled 0.00. When ready, click Export Active Display... under the File menu in the Science Workshop window.

4) Import the data into Excel. The time should be in column “A” and the amplitude in column “B”. Under the Tools menu of Excel will be Data Analysis...; click it, and choose Fourier Analysis. Select the data to be analyzed (the number of data points must be \(2^n = 2, 4, 8, 16, \ldots\) - choose the largest number possible up to 4096). Put the Fourier transform into column “C” (you’ll have to reformat the width of the column). Put the frequency in column “D”; the frequency is the row number times the sampling rate divided by the number of points used, and since the row number is the sampling rate times the time (column “A”), make column “D” = Product(sampling rate, sampling rate, 1/number points, A). Finally, put the absolute value of the Fourier transform into column “E”; under More Functions you’ll find IMABS, make column “E” = IMABS(C).

5) Make a graph of columns “D” vs. “E”. The peaks in the graph should give the original frequencies of the function generator.

Analysis

Print the graph (from Science Workshop) of the beats twice: choose the axes so that \(\tilde{f}\) can be easily determined from one plot, and \(f_{beat}\) can be easily determined from the other. Print the graph of the Fourier transform (from Excel), and label the frequencies of the peaks. How do the values of the frequencies as estimated using the Science Workshop graphs compare to the frequencies determined using the Fourier transform?

3.2 Doppler shift

Apparatus The apparatus needed for this experiment is a computer with a Pasco Science Workshop interface attached, a sound sensor and two photogates attached to the Science Workshop interface. You’ll also need a
master photogate timer (not attached to the Science Workshop interface), an airtrack and cart, and a battery powered, monochromatic sound source. The sources we have have a frequency of about 3500 Hz. Record the length of that part of the cart that passes through the master photogate timer. You’ll need this to calculate speed later.

**Procedure**

1) Set up the experiment. Connect the sound sensor and photogates to the Science Workshop interface box. Under **Sampling Options**, set the sampling rate to 10000 Hz and start and stop conditions. For a start condition, choose **Channel**, then **Digital 1**, then **Low (blocked gate)**; for a stop condition, choose **Channel**, then **Digital 2**, then **Low (blocked gate)**. Place photogate #2 as near to the sound sensor as possible, just make certain that the cart trips the photogate before the cart hits the bumper. Place photogate #1 about 1 m away from photogate #2. Place the master photogate timer somewhere in between.

2) Turn on the airtrack air source and sound source. Click the record button and slide the cart towards the sound sensor. Look at a graph of the sound amplitude vs. time. The amplitude should start off small and then increase. This is related to the loudness or volume of the sound. The sound will become louder as the sound source gets nearer the sound sensor. Verify that the frequency is not changing by using the fast Fourier transform tool built into Science Workshop. You can move the small window around to select different subsets of the data. Notice that the frequency doesn’t change with amplitude.

3) If the data in step 2) look good, you’re ready to record data. Click record, and reset the master photogate timer. Slide the cart towards the sound sensor to record moving data. Record the time from the master photogate timer: it will allow you to calculate the speed of the cart. As quickly as possible, click record again and take data with the cart stationary. Trip the photogates manually. Look at these data in graph windows, and if you’re happy with them, export the data for later analysis by *Excel*. Label the data files clearly.

4) Reverse the setup of the airtrack. Place photogate #2 as far from the sound sensor as possible, and photogate #1 about 1 m nearer. Keep the master photogate timer somewhere in between.

5) Turn on the airtrack air source and sound source. Click the record button and slide the cart away from the sound sensor. Do a quick analysis of these data as in step 2), but now the amplitude should start off high and drop as the cart moves farther away from the sound sensor.

6) If the data in step 5) look good, you’re ready to record data. Click record and reset the master photogate timer. Slide the cart away from the sound sensor to record moving data. Record the time from the master photogate timer: it will allow you to calculate the speed of the cart. As quickly as possible, click record again and take data with the cart stationary. Trip the photogates manually. Look at these data in graph windows, and if you’re happy with them, export the data for later analysis by *Excel*. Label the data files clearly.

7) Using the Fast Fourier Transform function of *Excel*, determine the frequency of the sound source for each of the 4 data sets taken above.

**Analysis**

Take the speed of sound in air to be 334 m/s.

For the data taken with the cart moving towards the sound sensor (and the stationary data taken immediately after), record the stationary frequency, the moving frequency, the speed and the expected moving frequency, using Eqn. 13. How do the actual and expected shifted frequencies compare? Is the shift in the correct direction?
For the data taken with the cart moving away from the sound sensor (and the stationary data taken immediately after), record the stationary frequency, the moving frequency, the speed and the expected moving frequency, using Eqn. 13. How do the actual and expected shifted frequencies compare? Is the shift in the correct direction?

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